Solution - Oscillating rope

A) It is evident from the figure that the curvature of the rope in the fundamental vibration is very small. It infers for a possibility to model the fundamental vibration as a swinging of a rigid uniform rod of length L about a pivot point at its end. The moment of inertia of the rod is:

$$I = \frac{mL^2}{3}$$

and the distance from the center-of-mass to the pivot point is:

$$b = L/2$$

Therefore, the frequency of the fundamental vibration is approximated as:

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{mgb}{I}} = \frac{1}{2\pi}\sqrt{\frac{2L}{3g}} \approx 0.61 \text{ Hz}$$

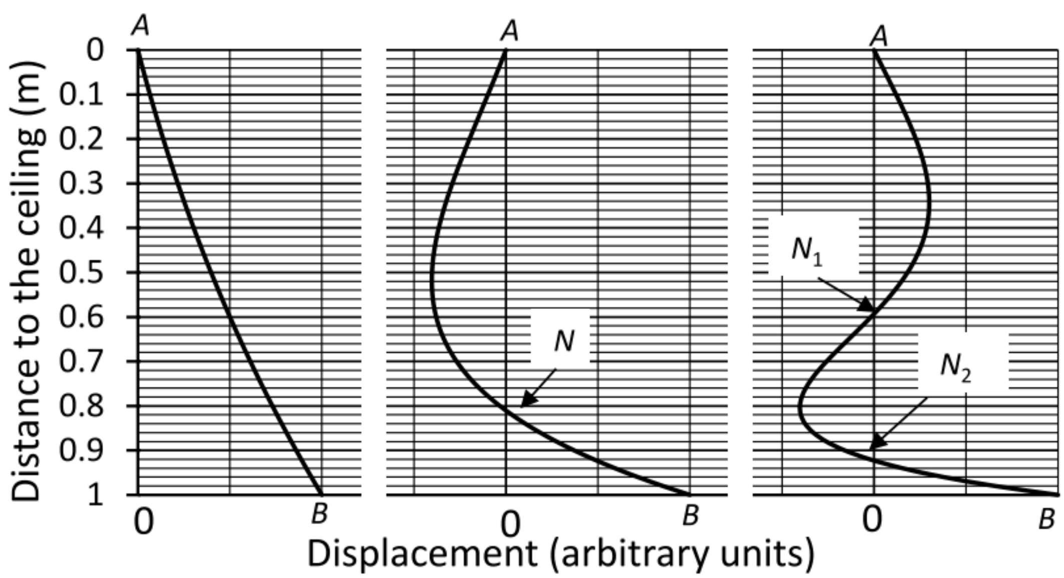
Correspondingly, the period of the fundamental vibration is:

$$T_1 = \frac{1}{2\pi}\sqrt{\frac{I}{mgb}} = \frac{1}{2\pi}\sqrt{\frac{3g}{2L}} \approx 1.6 \text{ s}$$

B) Whatever model for estimating of f_1 is being used, one may deduce on the basis of dimensionality arguments that the k-th natural frequency of the rope is:

$$f_k = c_k\sqrt{\frac{g}{L}}$$

where c_k is a dimensionless numeric coefficient depending on the consecutive mode number k only. Let A and B be the suspension point and the free end of the rope respectively, and N be the node on the rope for the second natural vibration (see the figure).



Since the node point is at rest (in the small-amplitude approximation), the vibration of the part NB could be considered as a fundamental vibration of a rope of length LNA about a suspension point N. Therefore:

$$f_2(L) \equiv f_1(L - NA)$$

Hence one may write:

$$\frac{f_2(L)}{f_1(L)} = \frac{f_1(L - NA)}{f_1(L)} = \sqrt{\frac{L}{L - NA}}$$

Since the absolute displacement is much smaller than the length of the rope, the distances could be measured in a vertical direction, to the ceiling, instead along the rope. Therefore, by taking $L = 1$ m, and $NA \approx 0.8$ m, we obtain:

$$\frac{f_2}{f_1} \approx 2.2$$

Similarly, the vibration of the part N_1B in the third eigenmode is equivalent to the second natural vibration of a rope of length $L - N_1A \approx 0.4$ m. In analogy to the first case:

$$f_3(L) \equiv f_2(L - NA)$$

and

$$\frac{f_3(L)}{f_2(L)} = \frac{f_2(L - N_1A)}{f_2(L)} = \sqrt{\frac{L}{L - N_1A}} \approx 1.6$$

Therefore:

$$\frac{f_3}{f_1} = \frac{f_2}{f_1} \times \frac{f_3}{f_2} \approx 3.5$$

Finally:

$$f_1 : f_2 : f_3 \approx 1 : 2.2 : 3.5$$

Part	Marking scheme	Points
A	States explicitly or realizes (proper drawing or notattions) the physical pendulum analogy	**1.0**
	Correct expression for the moment of inertia	**1.0**
	Determination of the position of the center of mass	**0.5**
	Correct formula for the period/frequency of a physical pendulum	**1.0**
	Calculates f_1 = 0.61 Hz with two significant digits	**0.5**
	Subtotal on A	**4.0**
B	States or realizes (proper graph or formula) that the vibration of the rope below a node point is similar to a lower order vibration of a shorter rope.	**1.0**
	Uses dimensionality arguments to argue that $f_k = c_k(g/L)^{1/2}$	**1.0**
	Applies similarity arguments to f_1 and f_2 and derives $f_2/f_1 = (L/L-NA)^{1/2}$	**1.0**
	Reads correctly NA from the graph	**0.5**
	Calculates f_2/f_1 = 2.2 to a precision of 2 significant digits	**0.5**
	Applies similarity arguments to f_3 and f_2 (or f_1) and derives $f_3/f_2 = (L/L-N_1A)^{1/2}$ or $f_3/f_1 = (L/L-N_2A)^{1/2}$	**1.0**
	Reads correctly N_1A (N_2A) from the graph.	**0.5**
	Calculates f_3/f_1 = 3.5 to a precision of 2 significant digits.	**0.5**
	Subtotal on B	**6.0**